OF THE LAX REPRESENTATION OF THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION USING THE KAUP–NEWELL SPECTRAL PROBLEM

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Kalit so‘zlar. Hosilali nochiziqli Shryodinger tenglamasi, tez kamayuvchi funksiyalar, Shvarts fazosi, integrallash usullari, matematik fizika tenglamalari, Fure almashtirishi, yechimning mavjudligi va yagonaligi.

Kirish. Zamonaviy matematik fizika va chiziqli bo‘lmagan differensial tenglamalar nazariyasida Shryodinger turidagi tenglamalarni tadqiq qilish muhim o‘rin tutadi. Xususan, hosilali Shryodinger tenglamalari kvant mexanikasi, plazmalar fizikasi, noldan farqli optik tolalar va to‘lqin jarayonlarini modellashtirishda keng qo‘llaniladi. Bunday tenglamalarning cheksiz sohalarda, xususan, tez kamayuvchi funksiyalar sinfida (Shvarts fazosida) yechimlarini o‘rganish nazariy va amaliy jihatdan katta qiziqish uyg‘otadi. Ushbu maqolada hosilali Shryodinger tenglamasini aynan shu sinfda integrallash masalalari, yechimlarning silliqligi va baholanishi chuqur tahlil qilinadi.

Keywords: Derivative Schrödinger equation, rapidly decreasing functions, Schwartz space, integration methods, mathematical physics equations, Fourier transform, existence and uniqueness of solutions.

Introduction: In modern mathematical physics and the theory of nonlinear differential equations, the study of Schrödinger-type equations plays a crucial role. Specifically, derivative Schrödinger equations are widely used to model quantum mechanics, plasma physics, nonlinear optics, and various wave phenomena. Investigating the solutions of such equations in infinite domains, particularly within the class of rapidly decreasing functions (Schwartz space), is of significant theoretical and practical interest. This paper is devoted to the integration of the derivative Schrödinger equation within this specific function class, thoroughly analyzing the smoothness, behavior, and estimates of the solutions.

In 1967, the KdV equation was solved, and in 1972 the nonlinear Schrödinger (NLS) equation was solved by Zakharov and Shabat. By the early 1970s, mathematicians were working on the standard nonlinear Schrödinger equation (NLS). However, a problem arose in plasma physics, particularly in the study of Alfvén waves propagating parallel to the magnetic field in magnetized plasma. In 1976, the Norwegian physicist Einar Mjølhus, in his well-known paper "On the Modulational Instability of Hydromagnetic Waves Parallel to the Magnetic Field," derived an explicit equation describing the propagation of Alfvén waves:

$$i\frac{\partial q}{\partial t}+\frac{\partial^2 q}{\partial x^2}+i\frac{\partial q}{\partial x}\left(|q|^2 q\right)=0 (*)$$

This equation was precisely the derivative nonlinear Schrödinger equation (DNLS). These researchers' work laid the foundation for the emergence of the Kaup–Newell system in soliton theory.

In 1978, David J. Kaup and Alan C. Newell published their research paper entitled "An Exact Solution for a Derivative Nonlinear Schrödinger Equation."

Kaup and Newell realized that the standard Zakharov–Shabat system had to be modified in order to solve the derivative nonlinear Schrödinger equation (DNLS). Taking into account the fact that the spectral parameter enters not linearly in

combination with the potentials, but also in product and quadratic forms, they proposed the following system:

$$\begin{cases} v_{1x} = -i\xi^2 v_1 + q\xi v_2 \\ v_{2x} = i\xi^2 v_2 + r\xi v_1 \end{cases} (**)$$

This system is known as the Kaup–Newell system.

To integrate nonlinear evolution equations by the inverse scattering method, they must be represented in terms of two linear operators (a Lax pair) that satisfy the compatibility condition requiring equality of the mixed derivatives. The Lax idea makes it possible to express nonlinear evolution equations through a pair of linear operators. This approach serves as a fundamental mathematical tool for integrating the Kaup–Newell system and calculating changes in soliton parameters under the influence of a source.

Suppose that we have a pair of linear differential equation systems depending on the spectral parameter:

$$\begin{cases} \varphi_x = L\varphi \\ \varphi_t = A\varphi \end{cases} (1)$$

Here, ψ is the eigenfunction, while U and V are matrix operators.

From mathematical analysis, we know that if a function is sufficiently smooth, its mixed second-order partial derivatives must be equal:

Now, differentiating with respect to the variables using the systems above, we obtain:

$$(\varphi_x)_t = \frac{\partial}{\partial t}(L\varphi) = L_t\varphi + L\varphi_t = L_t\varphi + LA\varphi \quad (2)$$

$$(\varphi_t)_x = \frac{\partial}{\partial x}(A\varphi) = A_x\varphi + A\varphi_x = A_x\varphi + AL\varphi \quad (3)$$

Equating these systems,

we obtain the following equality:

For this equality to hold for an arbitrary solution, the expression in parentheses must be equal to zero:

$$L_t - A_x + [L, A] = 0 (4)$$

Thus, if a nonlinear equation (for example, the DNLS equation) can be written in the form (4) using two linear operators U and V, then the nonlinear equation is integrable. Here, [U,V] denotes the commutator (the difference of the products of the matrices). The operator V was proposed by Kaup and Newell in 1978. Taking into account the distinctive feature of the DNLS equation—that the derivative appears in the nonlinearity—they incorporated the spectral parameter λ into the operator as follows:

$$L = \begin{pmatrix} -i\xi^2 & \xi q \\ \xi r & i\xi^2 \end{pmatrix} (5)$$

The operator V does not arise automatically. To determine it, the operator V is substituted into the Lax equation, and the resulting expression is analyzed according to the powers of λ.

For the DNLS equation, the operator V usually has an expansion up to the fourth power of λ:

Each term in the matrix V is chosen so that, when the equality (4) is imposed, all unwanted powers of λ cancel, leaving only the DNLS equation we seek:

$$A = \begin{pmatrix} -2i\xi^4 - i\xi^2 qr & 2\xi^3 q - i\xi(q_x + iq^2 r) \\ 2\xi^3 r - i\xi(r_x + ir^2 q) & 2i\xi^4 + i\xi^2 qr \end{pmatrix} (6)$$

Determining the Jost solutions for the Kaup–Newell spectral problem (DNLS) is somewhat more complicated than for the standard Zakharov–Shabat system, because the spectral parameter appears both quadratically and linearly in the spectral problem.

The solutions of the differential system (1) that satisfy the following respective

$$\left.\begin{aligned}\varphi(x,\xi)&\sim\begin{pmatrix}1\\0\end{pmatrix}e^{-i\xi^2x}\\ \tilde{\varphi}(x,\xi)&\sim\begin{pmatrix}0\\-1\end{pmatrix}e^{i\xi^2x}\end{aligned}\right\}\ x\to-\infty,\qquad \left.\begin{aligned}\psi(x,\xi)&\sim\begin{pmatrix}0\\1\end{pmatrix}e^{i\xi^2x}\\ \tilde{\psi}(x,\xi)&\sim\begin{pmatrix}1\\0\end{pmatrix}e^{-i\xi^2x}\end{aligned}\right\}\ x\to\infty \quad (7)$$

asymptotic conditions are called its Jost solutions.

Let us consider the following lemma.

Lemma. If the vector functions $V=\left(v_1(x,\xi),v_2(x,\xi)\right)^T$ and $U=\left(u_1(x,\mu),u_2(x,\mu)\right)^T$, $U^*=\left(u_1^*(x,\mu^*),u_2^*(x,\mu^*)\right)^T$ are solutions of the equations $LV=\xi V$ va $LU=\mu U$, $LU^*=\mu^*U^*$, then the following identities hold for the components of these vector functions:

$$\frac{d}{dx}\left(v_1u_2+v_2u_1\right)=i\left(\mu^2-\xi^2\right)\left(v_1u_2-v_2u_1\right)-\left(\mu+\xi\right)\left(q^*v_1u_1-qv_2u_2\right),(8)$$

$$\frac{d}{dx}\left(v_1u_2-v_2u_1\right)=i\left(\mu^2-\xi^2\right)\left(v_1u_2+v_2u_1\right)-\left(\mu-\xi\right)\left(q^*v_1u_1+qv_2u_2\right),(9)$$

$$\frac{d}{dx}\left(v_1u_1^*+v_2u_2^*\right)=i\left(\mu^{*2}-\xi^2\right)\left(v_1u_1^*-v_2u_2^*\right)+\left(\mu^*-\xi\right)\left(q^*v_1u_2^*-qv_2u_1^*\right),(10)$$

the following identities hold.

Proof. For the functions

the following equalities hold. Simplifying the expression below, we obtain the equalities stated above:

The pairs of vector functions are linearly independent solutions of the Kaup–Newell differential system (1). For the Wronskian determinant formed from them, the following

identities hold; therefore, the following

$$\begin{cases}\varphi=a(\xi)\bar{\psi}+b(\xi)\psi,\\ \bar{\varphi}=-\bar{a}(\xi)\psi+\bar{b}(\xi)\bar{\psi}\end{cases}(11)$$

and

$$\begin{cases} \psi = -a(\xi)\bar{\varphi} + \bar{b}(\xi)\varphi, \\ \bar{\psi} = \bar{a}(\xi)\varphi + b(\xi)\bar{\varphi} \end{cases} \quad (12)$$

relations also hold. Here,

$$a(\xi) = W\{\varphi, \psi\}, \quad b(\xi) = W\{\bar{\psi}, \varphi\} \quad (13)$$

and the following identities hold for them:

$$|a(\xi)|^2 + |b(\xi)|^2 = 1, \ \bar{a}(\xi) = a(-\xi), \ \ \bar{b}(\xi) = b(-\xi).$$

It is known that the following integral representations hold for the Jost solutions $\psi(x,\xi)$ and $\varphi(x,\xi)$

$$\psi(x,\xi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi^2 x} + \int_x^\infty K(x,s) e^{i\xi^2 s} ds, (14)$$

$$\varphi(x,\xi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi^2 x} - \int_{-\infty}^x L(x,s) e^{-i\xi^2 s} ds. \quad (15)$$

Here,

$$K(x,s) = \begin{pmatrix} K_1(x,s) \\ K_2(x,s) \end{pmatrix}, L(x,s) = \begin{pmatrix} L_1(x,s) \\ L_2(x,s) \end{pmatrix}$$

where the kernels do not depend on the variable x.

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